Understanding The Physics of Slingshots

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Abstract

In an effort to understand the physics of a slingshot I had to learn some algebra, some calculus, and some basic physics or the basics of classical mechanics. The most helpful book for an introductory crash course on these topics was the *"No Bullshit Guide to Math & Physics"* by Ivan Savov. If you struggled with math in school or dropped out like I did, this book will give you all you need to begin understanding your slingshots.

After the math has been understood, the next step is to read retired physicist, Bob Yeats', paper titled:

"Physical modeling of real-world slingshots for accurate speed predictions"

His paper is a gateway of sorts. The why's of what is needed to understand slingshots is covered very well, but the mathematical ways are not. At least that was my experience. So my hope is to help you by saving you the time it took me to build the foundation required to take this information in.

In his paper, Bob Yeats refers to a method of calculating the speed called leapfrog. Believe it or not, this is a technical term in the physics world. The reference (#11) he sites from his paper goes very deep into the way it works, and was beyond me until I read Savov's book and some Wikipedia items on the topic. I'll try to explain in my own words how all this works together but I am not the source of this information. Bob and his references are where the credit goes. My focus here is on where I struggled.

1 Math

I'll start by going into the math I had to learn and know to begin my overall understanding:

1.1 Algebra

This is a tough one because I learned algebra by programming when I was a young man. My learning of it was unorthodox to say the least. But calculus is unusable without it. I picked up a copy of *"Algebra II for Dummies"* to get it locked in. But for the purposes of this learning process I found that there was not much direct algebra needed to get this nailed down. However, I could be taking some things for granted here.

1.2 Calculus

I don't know about you but I found learning calculus to be frustrating at first and then it was just too cool. I picked up a copy of *"Calculus for Dummies"* which was good but the *"How to Ace Calculus the Streetwise Guide"* was better. I must say the *"No Bullshit Guide to Math & Physics"* was the best bang for the buck... period.

In Calculus there are two basic concepts and various branches from them, these are Derivation and Integration. It is very important to understand the basis of these two practices. Calculus is all about *the change of things*. In our case we are concerned with the changes in acceleration as our ammo reaches its final speed.

1.2.1 The Derivative

If a mathematical algorithm is a function, the derivative is a function that represents the change of that function. If you are cruising in your car at 30 miles per hour the result of the derivative of your movement will be 1. If you are cruising at 60 mph the resulting derivative will again be 1. This is because there is no change happening.

The mathematical definition of the derivative is following:

$\frac{dy}{dx}f(x) = f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

The little tick ($'$) next to the $f(x)$ means derivative of $f(x)$. But in physics a dot above the f like this $\dot{f}(x)$ is used.

With the definition above, if we were to say that your speed is increasing because you have the pedal to the floor and your hotrod is now *accelerating* at a rate of 10 miles per hour every hour, because your pulling a large boat uphill in the mountains with your funny car, we would say that your acceleration can be expressed like this:

$a = 10$ miles per hour... per hour ($10^{mi/h^2}$)

If $v = at$ (from a dead stop), where $a =$ acceleration and $t =$ time lapsed in hours, we could say that in 3 hours, $v = (10^{mi/h^2} \cdot 3)$ is equal to 30 miles per hour at the 3 hour mark. This may be overly simple but what I am showing here is that a is the rate of increase in speed even though it is the a does not change over time it shows the change in v over time. In other words acceleration is the derivative of velocity.

1.2.2 The Integral

If *acceleration* is the derivative of *velocity*, it can be said that *velocity* is the integral of *acceleration.* This is sometimes called the *anti-derivative.* Now velocity is the derivative of position. Meaning that if you start at a given point and go straight for a given amount of time at a given velocity after that time lapses you will be in a new position. You can determine this by integrating your velocity (which you may have done before not realizing it).

Going back to the above example, if velocity is the integral of position we can see it like this:

$p = vt$, so when v is 30 and t is 3 hours that passed, the total distance travled is $p = 30 \cdot 3 = 90$ miles. But wait, in our earlier example we had acceleration so velocity was changing. This means we need the integral of the integral of acceleration.

$p(t) = \int \dot{p}(t)dt = \frac{1}{2}at^2$

This formula incorporates the rate of change in velocity over the course of time given our fixed velocity. That "S" looking thing ($\int$) basically represents the sum of the parts that were the derivative parts of the original formula.

Using the above formula we can determine, from our dead stop, how far we traveled with our boat in 3 hour accelerating at $10^{mi/h^2}$.

$p = \frac{1}{2}at^2 = \frac{1}{2} \cdot 10 \cdot 3^2 = 45$ miles from where we started.

So, what do we do when acceleration changes?

2 Physics - What we care of it

Physics is about measurement and calculation of the behavior of physical objects. From the late 1600s until the 1900s there was only one flavor of physics. It was primarily based on Issac Newton's discoveries as he observed and measured the things around us and the effects of gravity on them. Newton established three basic findings that are an outline of what we often call Newtonian Physics also known as Classical Mechanics. These were said to be laws and are called Newtons Laws of Physics. They explain the behavior of many actions and results of those actions, but fail to explain gravity. In around 1920 Albert Einstein hit the seen with the Theory of Relativity ($E = mc^2$) in an effort to explain gravity and the behavior of light. This soon lead to the disovery (or development) of Quantum Mechanics (aka Quantum Physics). We don't have plans to hurl uranium with our slingshots and we haven't approached the speed of light yet, so we only need to make use of Classical Mechanics which I will call Physics or Mechanics here on out.

2.1 Laws of Physics

As basic as these laws sound is as basic as the ideas are that are behind the math we will be using. But don't get me wrong the math is still tricky.

Newton's First Law basically says that an object in motion stays in motion unless acted upon by an opposing force (like friction or wind drag) and an object at rest stays at rest unless something messes with it. This is where the idea of *inertia* (from Galileo) applies.

Newton's Second Law is usually expressed as Force is equal to mass times acceleration. It and is mathematically expressed as $F = ma$. This is where we will focus our study.

Newton's Third Law: For every action there is an equal and opposite reaction. Which is where impact force comes into play.

2.2 Formulas, Methods and Concepts we will use here

2.2.1 Acceleration

$F = ma$, where F is force, m is mass (not weight) measured in Newtons, and a is the rate of acceleration in meters per second per second (m/s^2). In slingshot land we already have access to the value of Force, which I will describe below. Mass is the weight of our ammo in kilograms times the acceleration of gravity (about 9.8 m/s^2 on Earth). With those two we can gather acceleration with a little algebra.

$a = \frac{F}{m}$

We can rephrase this as the derivitive of velocity is equal to acceleration:

$\dot{v}(t) = a = \frac{F}{m}$

2.2.2 Velocity

The derivative of the of position is its velocity which is equal to its current velocity plus the integral of its acceleration for a time:

$\dot{x}(t) = \int \ddot{x}(t)dt = v + \frac{F}{m}t$

Although velocity is what we are looking for from our slingshot band configuration. We also need to know the position of the ammo in relation to the stretch of the bands. So we have to keep going.

2.2.3 Displacement

The position is equal to the integral of the velocity which is also the integral of the integral of acceleration:

$x(t) = \int \dot{x}(t)dt = x + vt + \frac{F}{2m}t^2$

Dropping the fancy stuff we have $x = x + vt + \frac{F}{2m}t^2$

Now we have to gather as much information about our bands as possible. Once this is done properly you can build theoretical slingshots without so much trial and error.

2.3 Gathering the slingshot characteristics

2.3.1 Retraction Force

The main information about your bands is the retraction reaction. The stretch is good to know for determining if you have the strength to pull the bands back (comfortably). The video method used by Bob Yeats is probably the most cost effective. This is where you attach a digital scale *with no spring*(I used a digital fish scale), and a measuring tape of sorts within the view of a camera. Could be your phone or a webcam. Smoothly stretch to your maximum stretch length (not the band's but in relation to the movement of your arms as you reach your full comfort length, i.e. to your chest). As you stretch keep it a pace that is within the limits of your camera. If you go too fast it will blur the image frames. So you stretch to your limit wait about the time you take to aim then begin the retraction.

Using this process I gathered data at each inch. I actually had to go slower than I wanted to due to the refresh rate of my scale.

Using the retraction these numbers you can establish a "factor" for your bands that will allow you to determine or predict the characteristics of other band dimensions.

2.3.2 Establishing Mass

I bought an Etekcity 500gram multiunit scale for $10 on amazon.com for weighting my ammo. Mostly because I was casting my own from lead in different sizes. But I prefer steel because I make my slingshots with magnets to hold ammo. This scale allows for calibration so I also bought a calibrating weight of 500 grams(which the instructions called for). The scale shows weight in grams, grains, oz, kt, and stuff I've never heard of. It is so sensetive that I have to shut off the ceiling fan before use it so the pressure of the wind is not measured. Ok, enough of the scale.

When factoring in weight you may have to dice up a test band set. You'll need the weight of the following:

- Ammo
- Pounch and rubber knots
- Band's stretching portion

Retraction is more important. Taper plays a small part. At least an inch resolution of stretch. Smoother retraction is better that jerky retraction. From full pull to zero tension as fast as reasonably possible. Kg or lbs must be convertet to Newtons of force. The length of stretch at each force point must be known. Stretch length is zero at zero tension.

2.4 Leapfrog Intergration

The leapfrog process referred to in Bob's document is used to take measurements and accumulate the calculated results into a final answer. For us the result is the final velocity.

It is called the leapfrog method because the calculation points jump over each other. In our case we need to satify the leapfrog method of calculating by making an initial calcuation for the start of the process and then as x as our position and v as our velocity we calculate at x_1 then $v_{\frac{1}{2}}$ then x_2 then $v_{\frac{3}{2}}$ and so on.

So the reason we need the leapfrog method is because the force on the ammo changes as the bands retract. This means that the acceleration changes too. That change does not correlate to a simple formula but rather to your retraction force measurements. Leapfrog is just what we need to deal with this change in force. I am a computer programmer so that is how I tend to think. I will demonstrate the calculations the only way I know how right now, in a program.

I am writing a web page that demostrates the leapfrog method in Java Script. You can right-click and view source to see the code behind these calculations:
http://busysteve.com/slingshot_calc.html

A video of my first measurement can be found here:
https://www.youtube.com/watch?v=u7kDAvgPjj8

3 Closing

The bands are cut from Theraband Gold and are tapered from 1 inch to $\frac{1}{2}$ inch, with two on each side. The pounch weight with the knots is .0035 kilograms and the ammo tested was steel $\frac{5}{8}$ inch ball and $\frac{1}{2}$ inch ball. The stretch length was 22 inches from no tension.

My calculated results where as follows:

- steel $\frac{5}{8}$" ball = 145.1679 fps
- steel $\frac{1}{2}$" ball = 159.256 fps

My Chrony mesurements for average of three shots each was:

- steel $\frac{5}{8}$" ball @ 147.2 fps
- steel $\frac{1}{2}$" ball @ 162.8 fps

I don't know about you but I am quite pleased with the results.

The measurement of bands' retraction yeilded the following after converting to Newtons:

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62.4683605
60.3108975
56.3882375
53.2501095
49.9158485
47.268053
45.9931885
44.8163905
41.4821295
38.9324005
36.7749375
35.500073
34.323275
31.6754795
29.223817
26.3798885
23.6340265
20.201699
16.3771055
12.6505785
8.5317855
5.1975245
0.0
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References

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